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Non-linear relations of anisotropic elasticity and a particular postulate of isotropy ☆

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Abstract

A general approach to the construction of six-dimensional images of strain processes is proposed with the introduction of a vector basis which, in special cases, is identical to the well-known bases of A. A. Il'yushin, V. V. Novozhilov and Ye. I. Shemyakin and S. A. Khristianovich. The analysis of the properties of materials is based on the use of the concept of characteristic elastic states, which was introduced in the papers of J. Rychlewski. In the case of an isotropic material and four types of anisotropic materials belonging to the cubic, hexagonal, trigonal and tetragonal systems, characteristic subspaces, corresponding to the multiple eigenvalues of the elasticity tensor are defined in a six-dimensional space. In accordance with Hooke's law, the components of the stress and strain vectors in these subspaces preserve their axial alignment for any of their orthogonal transformations. The particular postulate of isotropy, formulated by Il'yushin, is therefore satisfied by definition within the framework of isotropic characteristic subspaces for linear elastic materials. An extension of the particular postulate to strain processes in non-linear anisotropic materials is proposed, on the basis of which a general form of constitutive relations is obtained containing a minimum number of experimentally determinable material functions.

1. Six-dimensional images of the changing processes of symmetric tensors

The relation between strains and stresses takes its simplest form when their six-dimensional vector images[1,2] are considered which enables one to reduce the rank of the tensor quantities occurring in the relations.

In the Cartesian basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ ($\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$), the strain tensor is represented by the dyadic expansion

$$\boldsymbol{\varepsilon} = \varepsilon_{ij}\mathbf{e}_i\mathbf{e}_j; \quad \varepsilon_{ij} = \varepsilon_{ji}, \quad i, j = 1, 2, 3 \tag{1.1}$$

We associate the six-dimensional vector

$$\mathbf{э} = э_\alpha \mathbf{a}_\alpha, \quad \alpha = 1, 2, \ldots, 6 \tag{1.2}$$

with the tensor (1.1) in such a way that the contraction of the tensor $\boldsymbol{\varepsilon}$ and the square of the length of the vector $э$ are equal, that is $э \cdot э = \boldsymbol{\varepsilon} \cdot\cdot \boldsymbol{\varepsilon}$. In the general case, the relation between the components ε_{ij} and $э_\alpha$ is defined by the linear

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operator β in the form

$$\varepsilon_{ij} = \beta_{ij}^{\alpha} э_{\alpha}, \quad э_{\alpha} = \beta_{\alpha}^{ij} \varepsilon_{ij} \tag{1.3}$$

It follows from these relations that the operator β satisfies the symmetry conditions

$$\beta_{ij}^{\alpha} = \beta_{ji}^{\alpha}, \quad \beta_{\alpha}^{ij} = \beta_{\alpha}^{ji}$$

the orthogonality conditions

$$\sum_{\alpha=1}^{6} \beta_{\alpha}^{ii2} = 1, \quad \sum_{\alpha=1}^{6} 2\beta_{\alpha}^{ij2} = 1 \ (i \neq j), \quad \sum_{\alpha=1}^{6} \beta_{\alpha}^{ij} \beta_{\alpha}^{kl} = 0 \ (i \neq k, j \neq l)$$

$$\sum_{i,j=1}^{3} \beta_{ij}^{\alpha 2} = 1, \quad \sum_{i,j=1}^{3} \beta_{ij}^{\alpha} \beta_{ij}^{\beta} = 0 \ (\alpha \neq \beta)$$

and, also, the conditions

$$\beta_{\alpha}^{ij} \beta_{ij}^{\beta} = \delta_{\alpha}^{\beta}, \quad \beta_{ij}^{\alpha} \beta_{\alpha}^{kl} = \delta_i^k \delta_j^l$$

from which it follows that the operator β is assigned by a single table. We require that the six-dimensional space with the basis $\mathbf{a}_\alpha$ ($\mathbf{a}_\alpha \cdot \mathbf{a}_\beta = \delta_{\alpha\beta}$) should separate into two independent three-dimensional subspaces. In one of them (the diagonal one), a three-dimensional vector is made to correspond to the diagonal components of the tensor and, in the other one of them (the shear one), a three-dimensional vector is made to correspond to the mixed components. In this case, only the components β_{ii}^{α}, for which $\boldsymbol{\alpha} = 1, 2, 3$ and $i = 1, 2, 3$ are non-zero components of the operator β and, also, $\beta_{12}^4 = \beta_{23}^5 = \beta_{31}^6 = 1/\sqrt{2}$.

A change in the orientation of the vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ with respect to the basis of the material space $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ enables one to construct a vector image with the different components $э_1$, $э_2$, $э_3$ for one and the same tensor $\boldsymbol{\varepsilon}$. The components $э_4 = \sqrt{2}\,\varepsilon_{12} э_5 = \sqrt{2}\,\varepsilon_{23}$, $э_6 = \sqrt{2}\,\varepsilon_{31}$ do not change in this case. On choosing $\beta_{11}^1 = \beta_{22}^2 = \beta_{33}^3 = 1$ and the non-diagonal components as being equal to zero, we obtain the principal basis $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. Putting

$$\beta_{11}^1 = \beta_{22}^1 = \beta_{33}^1 = \frac{1}{\sqrt{3}}, \quad \beta_{11}^2 = \beta_{22}^2 = -\frac{1}{2}\beta_{33}^2 = -\frac{1}{\sqrt{6}}, \quad \beta_{11}^3 = -\beta_{22}^3 = \frac{1}{\sqrt{2}}, \quad \beta_{33}^3 = 0$$

we obtain Il'yushin's vector basis[2] (with the traditional numbering of the vectors)

$$\mathbf{i}_0 = \frac{1}{\sqrt{3}}(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3),\ \mathbf{i}_1 = \frac{1}{\sqrt{6}}(2\mathbf{a}_3 - \mathbf{a}_1 - \mathbf{a}_2),\ \mathbf{i}_2 = \frac{1}{\sqrt{2}}(\mathbf{a}_1 - \mathbf{a}_2),\ \mathbf{i}_3 = \mathbf{a}_4,\ \mathbf{i}_4 = \mathbf{a}_5,\ \mathbf{i}_5 = \mathbf{a}_6 \tag{1.4}$$

and Novozhilov's basis[3] which is analogous to it. Putting

$$\beta_{11}^1 = \beta_{33}^1 = \beta_{11}^3 = -\beta_{33}^3 = \frac{1}{\sqrt{2}}, \quad \beta_{22}^1 = \beta_{11}^2 = \beta_{33}^2 = \beta_{22}^3 = 0, \quad \beta_{22}^2 = 1$$

we obtain the Shemyakin - Khristianovich basis[4] in the form

$$\mathbf{h}_1 = \frac{1}{\sqrt{2}}(\mathbf{a}_1 + \mathbf{a}_3), \quad \mathbf{h}_2 = \mathbf{a}_2, \quad \mathbf{h}_3 = \frac{1}{\sqrt{2}}(\mathbf{a}_1 - \mathbf{a}_3), \quad \mathbf{h}_4 = \mathbf{a}_4, \quad \mathbf{h}_5 = \mathbf{a}_5, \quad \mathbf{h}_6 = \mathbf{a}_6$$

Suppose the correspondence between $\boldsymbol{\varepsilon}$ in a three-dimensional space and the vector э in the six-dimensional space is fixed, that is, the operator β is chosen and it does not subsequently change. In turn, the material basis of the three-dimensional space $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ is subjected to an orthogonal transformation with the operator $\mathbf{Q} = q_{ij}\mathbf{e}_i\mathbf{e}_j$, where

$$\mathbf{e}_i' = \mathbf{e}_i \cdot \mathbf{Q} = \mathbf{Q}^T \cdot \mathbf{e}_i, \quad \mathbf{Q}^T = \mathbf{Q}^{-1} \tag{1.5}$$

An orthogonal transformation of the basis with the operator **m** corresponds to the transformation (1.5) in six-dimensional space. A relation between the components of the operators **Q** and **m** can be obtained from relations (1.3) and has the form

$$m_{\alpha\beta} = \beta^{ij}_{\beta}\beta^{\alpha}_{kl}q_{ik}q_{lj} \tag{1.6}$$

The matrices of the operators **m** for the Il'yushin and Novozhilov vector bases were presented earlier in Refs. 1,3,5.

2. Properties of linearly elastic materials

Suppose the stresses and strains are connected by the linear tensor relations

$$\mathbf{S} = \mathbf{N}\cdot\cdot\boldsymbol{\varepsilon}, \quad \text{on} \quad S_{ij} = N_{ijkl}\varepsilon_{kl} \tag{2.1}$$

where the fourth rank elasticity tensor $\mathbf{N} = N_{ijkl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l$ has components which are symmetric with respect to the pairs of subscripts: $N_{ijkl} = N_{ijkl} = N_{ijlk} = N_{klij}$.

We will denote the six-dimensional image of the stress tensor **S** by $\boldsymbol{\sigma}$. The components σ_α and S_{ij} are connected by relations (1.3). The equality

$$\boldsymbol{\sigma} = \mathbf{n}\cdot\mathbf{э}, \quad \text{on} \quad \sigma_\alpha = n_{\alpha\beta}э_\beta \tag{2.2}$$

where $\mathbf{n} = n_{\alpha\beta}\mathbf{a}_\alpha\mathbf{a}_\beta$ is a second-rank symmetric tensor, the components of which, when β is fixed, are connected with the components of the elasticity tensor by the one-to-one relations

$$N_{ijkl} = \beta^{\alpha}_{ij}n_{\alpha\beta}\beta^{kl}_{\beta} \text{ and } n_{\alpha\beta} = \beta^{ij}_{\alpha}N_{ijkl}\beta^{\beta}_{kl} \tag{2.3}$$

corresponds to the linear relation between the stress and strain tensors (2.1) in six-dimensional space.

In the case of anisotropic materials, the structure of the tensors **N** and **n** is determined by the symmetry of the properties which is characteristic of them. The structure of the tensor **N** is known[3,6,7] for anisotropic materials belonging to different crystallographic systems and, on the basis of this, the matrices of the components of the tensor **n** in the principal basis and in the basis (1.4) have been obtained in Ref. 5.

3. Characteristic elastic states of materials

We will consider the classification of **n**-tensors by analysing their expansions in characteristic bases. According to Rychlewski's definition,[8] a strain tensor $\boldsymbol{\varepsilon}_\alpha$ which is such that

$$\mathbf{N}\cdot\cdot\boldsymbol{\xi}_\alpha = \lambda_\alpha\boldsymbol{\xi}_\alpha, \quad \boldsymbol{\xi}_\alpha = (\boldsymbol{\varepsilon}_\alpha\cdot\cdot\boldsymbol{\varepsilon}_\alpha)^{-1/2}\boldsymbol{\varepsilon}_\alpha \tag{3.1}$$

is called a characteristic tensor of the operator **N** (a characteristic elastic state).

In six-dimensional space, condition (3.1) takes the form

$$\mathbf{n}\cdot\boldsymbol{\omega}_\alpha = \lambda_\alpha\boldsymbol{\omega}_\alpha \tag{3.2}$$

where the vectors $\boldsymbol{\omega}_\alpha$ are the six-dimensional images of the tensors $\boldsymbol{\xi}_\alpha$.

The expansion of a tensor **n** in a characteristic base depends on the existence and structure of multiple eigenvalues λ_α. If all six eigenvalues are different, the expansion has the form

$$\mathbf{n} = \lambda_1\boldsymbol{\omega}_1\boldsymbol{\omega}_1 + \lambda_2\boldsymbol{\omega}_2\boldsymbol{\omega}_2 + \lambda_3\boldsymbol{\omega}_3\boldsymbol{\omega}_3 + \lambda_4\boldsymbol{\omega}_4\boldsymbol{\omega}_4 + \lambda_5\boldsymbol{\omega}_5\boldsymbol{\omega}_5 + \lambda_6\boldsymbol{\omega}_6\boldsymbol{\omega}_6 = \sum_{\alpha=1}^{6}\lambda_\alpha\boldsymbol{\Omega}_\alpha$$

where $\boldsymbol{\Omega}_\alpha = \boldsymbol{\omega}_\alpha\boldsymbol{\omega}_\alpha$ is the tensor of the characteristic basis.

If a root λ_α has a multiplicity k, the part of the expansion of the tensor **n** corresponding to it has the form

$$\mathbf{n}_{(\alpha)} = \lambda_\alpha(\boldsymbol{\omega}_\alpha\boldsymbol{\omega}_\alpha + \boldsymbol{\omega}_{\alpha+1}\boldsymbol{\omega}_{\alpha+1} + \ldots + \boldsymbol{\omega}_{\alpha+k-1}\boldsymbol{\omega}_{\alpha+k-1}) = \lambda_\alpha\boldsymbol{\Omega}_\alpha$$

where $\boldsymbol{\Omega}_\alpha = \boldsymbol{\omega}_\alpha\boldsymbol{\omega}_\alpha + \boldsymbol{\omega}_{\alpha+1}\boldsymbol{\omega}_{\alpha+1} + \ldots + \boldsymbol{\omega}_{\alpha+k-1}\boldsymbol{\omega}_{\alpha+k-1}$ is the basis tensor corresponding to the root λ_α.

In the general case, we can represent the expansion of a tensor **n** in a characteristic basis in the form

$$\mathbf{n} = \sum_{\alpha=1}^{n} \lambda_\alpha \mathbf{\Omega}_\alpha \tag{3.3}$$

where n is the number of different eigenvalues of the tensor **n**.

A characteristic subspace, the size of which is equal to the multiplicity of the corresponding root, is associated with each characteristic tensor $\mathbf{\Omega}_\alpha$. Any vector, belonging to a characteristic subspace of the operator **n** is its eigenvector.

The tetravalent basis tensor of a three-dimensional space $\mathbf{\Lambda}_\alpha$, which corresponds to the basis tensor $\mathbf{\Omega}_\alpha$ of the six-dimensional space, is obtained using transformation (2.3). In this case, on the basis of representations (3.3), the expansion of the elastic operator **N** in the characteristic basis has the form

$$\mathbf{N} = \sum_{\alpha=1}^{n} \lambda_\alpha \mathbf{\Lambda}_\alpha \tag{3.4}$$

The tensor **N**, corresponding to a certain type of anisotropy, is invariant with respect to subgroups of the orthogonal transformations **Q** of the material basis $\mathbf{e}_i$. Its image **n** is invariant with respect to the subgroup of orthogonal transformations **m**, which are related to **Q** by expression (1.6). It is significant that the tensor **n** can be invariant with respect to a wider subgroup than the subgroup **m**, which is generated by an orthogonal transformation. In particular, the basis tensor $\mathbf{\Omega}_\alpha$ is invariant with respect to the group of characteristic orthogonal transformations $\mathbf{m}_{(\alpha)}$, if $\mathbf{m}_{(\alpha)} \cdot \mathbf{\Omega}_\alpha \cdot \mathbf{m}_{(\alpha)}^{-1} = \mathbf{\Omega}_\alpha$. In this case, the characteristic orthogonal transformations do not change the size of the characteristic subspace, that is, any vector $\boldsymbol{\omega}' = \mathbf{m}_{(\alpha)} \cdot \boldsymbol{\omega}$ is decomposable in the characteristic basis $\boldsymbol{\omega}_\alpha, \omega_{\alpha+1}, \ldots, \omega_{\alpha+k-1}$.

We will now present expressions for the characteristic basis tensors $\mathbf{\Omega}_\alpha$ characterizing materials with different types of anisotropy.

Isotropic materials have two characteristic subspaces: a one-dimensional subspace with a basis tensor $\mathbf{\Omega}_1 = \mathbf{i}_0\mathbf{i}_0$ and a five-dimensional subspace with a basis tensor $\mathbf{\Omega}_2 = \mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2 \ldots + \mathbf{i}_5\mathbf{i}_5$; here, $\mathbf{i}_\alpha$ ($\alpha = 1, \ldots, 5$) are the basis vectors (1.4). The elasticity tensor in the six-dimensional space is represented in the form

$$\mathbf{n} = \lambda_1 \mathbf{\Omega}_1 + \lambda_2 \mathbf{\Omega}_2 \tag{3.5}$$

where $\lambda_1 = 3K$, $\lambda_2 = 2G$ are elastic moduli.

Cubic materials have three characteristic subspaces: a one-dimensional subspace with a basis tensor $\mathbf{\Omega}_1 = \mathbf{i}_0\mathbf{i}_0$, a two-dimensional subspace with a basis tensor $\mathbf{\Omega}_2 = \mathbf{i}_1\mathbf{i}_1 + \mathbf{i}_2\mathbf{i}_2$ and a three-dimensional subspace with a basis tensor $\mathbf{\Omega}_3 = \mathbf{i}_3\mathbf{i}_3 + \mathbf{i}_4\mathbf{i}_4 + \mathbf{i}_5\mathbf{i}_5$.

Hexagonal and trigonal materials have four characteristic subspaces: two one-dimensional subspaces with bases $\mathbf{\Omega}_1 = \boldsymbol{\omega}_1\boldsymbol{\omega}_1$ and $\mathbf{\Omega}_2 = \boldsymbol{\omega}_2\boldsymbol{\omega}_2$, where the eigenvectors are connected with the basis vectors by the relations

$$\boldsymbol{\omega}_1 = \mathbf{i}_0 \cos\varphi + \mathbf{i}_1 \sin\varphi, \quad \boldsymbol{\omega}_2 = -\mathbf{i}_0 \sin\varphi + \mathbf{i}_1 \cos\varphi$$

and the angle φ is expressed in terms of the components $n_{\alpha\beta} = \mathbf{i}_\alpha \cdot \mathbf{n} \cdot \mathbf{i}_\beta$

$$\operatorname{tg}\varphi = \frac{2n_{01}}{n_{00} - n_{11} + \sqrt{(n_{00} - n_{11})^2 + 4n_{01}^2}}$$

and two-dimensional characteristic subspaces with bases $\mathbf{\Omega}_3 = \mathbf{i}_2\mathbf{i}_2 + \mathbf{i}_3\mathbf{i}_3$ and $\mathbf{\Omega}_4 = \mathbf{i}_4\mathbf{i}_4 + \mathbf{i}_5\mathbf{i}_5$ in the case of hexagonal materials, and bases $\mathbf{\Omega}_3 = \boldsymbol{\omega}_3\boldsymbol{\omega}_3 + \boldsymbol{\omega}_5\boldsymbol{\omega}_5$ and $\mathbf{\Omega}_4 = \boldsymbol{\omega}_4\boldsymbol{\omega}_4 + \boldsymbol{\omega}_6\boldsymbol{\omega}_6$ in the case of trigonal materials, and, in the latter case, the orientation of the characteristic vectors with respect to basis (1.4) is determined by the relations

$$\boldsymbol{\omega}_3 = \mathbf{i}_2 \cos\psi + \mathbf{i}_4 \sin\psi, \quad \boldsymbol{\omega}_4 = -\mathbf{i}_2 \sin\psi + \mathbf{i}_4 \cos\psi$$

$$\boldsymbol{\omega}_5 = \mathbf{i}_3 \cos\psi + \mathbf{i}_5 \sin\psi, \quad \boldsymbol{\omega}_6 = -\mathbf{i}_3 \sin\psi + \mathbf{i}_5 \cos\psi$$

$$\operatorname{tg}\psi = \frac{2n_{24}}{n_{22} - n_{44} + \sqrt{(n_{22} - n_{44})^2 + 4n_{24}^2}}$$

Tetragonal materials have five characteristic subspaces: four one-dimensional spaces with bases $\mathbf{\Omega}_1 = \boldsymbol{\omega}_1\boldsymbol{\omega}_1$, $\mathbf{\Omega}_2 = \boldsymbol{\omega}_2\boldsymbol{\omega}_2$, $\mathbf{\Omega}_3 = \mathbf{i}_2\mathbf{i}_2$, $\mathbf{\Omega}_4 = \mathbf{i}_3\mathbf{i}_3$, where the eigenvectors $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$ are connected with the basis vectors by the same relations as in the case of hexagonal or trigonal materials, and a unique two-dimensional characteristic subspace with basis $\mathbf{\Omega}_5 = \mathbf{i}_4\mathbf{i}_4 + \mathbf{i}_5\mathbf{i}_5$.

The invariance of the operators $\mathbf{n}$ with respect to the characteristic transformations follows from the expressions which have been given for characteristic basis tensors. For example, the operator for an isotropic material (3.5) is invariant with respect to rotation and reflection of the six-dimensional basis with respect to the vector $\mathbf{i}_0$. This transformation can be treated as a rotation of Il'yushin's basis (1.4) about the $\mathbf{i}_0$ axis, which is orthogonal to the deviator of the plane $\mathbf{i}_1$, $\mathbf{i}_2$. At the same time, the angle of the form of the deformed state changes in proportion to the angle of rotation. However, the change in this angle has no effect on the linear relation between the stresses and strains described by Hooke's law.

Hence, the operator $\mathbf{n}$ of linearly elastic materials satisfies the condition of invariance with respect to the subgroup of characteristic orthogonal transformations. This condition is a consequence of the linearity of the relation between the stress and strain tensors and is determined by the uniqueness of this relation.

Using expansion (3.3), we can represent Hooke's law in the form

$$\boldsymbol{\sigma} = \sum_{\alpha=1}^{n} \lambda_\alpha \boldsymbol{э}_{(\alpha)} \tag{3.6}$$

where $э_{(\alpha)} = э \cdot \mathbf{\Omega}_\alpha$ is the vector projection of the strain image onto the characteristic subspace corresponding to λ_α. On formally putting all $\lambda_\alpha = 1$ in relation (3.6), we obtain spectral expansions of the vector of the deformations corresponding to the different types of anisotropy:

$$\boldsymbol{э} = \boldsymbol{э}_{(1)} + \boldsymbol{э}_{(2)} + \ldots + \boldsymbol{э}_{(n)} \tag{3.7}$$

In the case of an isotropic material, from relations (3.5) and (3.7) when $n = 2$ we obtain

$$\boldsymbol{э} = \boldsymbol{э}_{(1)} + \boldsymbol{э}_{(2)}$$

where $э_{(1)} = (э \cdot \mathbf{i}_0)\mathbf{i}_0$ is the image of the spherical component of the strain tensor and $э_{(2)} = э \cdot \mathbf{\Omega}_2$ is the image of the deviator component. We obtain the spectral expansion of the strains in the case of hexagonal materials in the form (3.7) for $n = 4$ and the components $э_{(\alpha)}$ have the following expansions in basis (1.5)

$$\boldsymbol{э}_{(1)} = (э_0\cos\varphi + э_1\sin\varphi)(\mathbf{i}_0\cos\varphi + \mathbf{i}_1\sin\varphi)$$

$$\boldsymbol{э}_{(2)} = (-э_0\sin\varphi + э_1\cos\varphi)(-\mathbf{i}_0\sin\varphi + \mathbf{i}_1\cos\varphi)$$

$$\boldsymbol{э}_{(3)} = э_2\mathbf{i}_2 + э_3\mathbf{i}_3, \quad \boldsymbol{э}_{(4)} = э_4\mathbf{i}_4 + э_5\mathbf{i}_5$$

The concept of spectral expansions of strains was introduced in Refs. 9,10. The expansion of the strain vector for a hexagonal material in basis (1.4) has the form[9,10]

$$\boldsymbol{э} = \mathbf{p}_{(1)} + \mathbf{p}_{(2)} + \mathbf{p}_{(3)} + \mathbf{p}_{(4)}$$

$$\mathbf{p}_{(1)} = \frac{1}{3}(\sqrt{2}э_0 - э_1)(\sqrt{2}\mathbf{i}_0 - \mathbf{i}_1), \quad \mathbf{p}_{(2)} = \frac{1}{3}(э_0 + \sqrt{2}э_1)(\mathbf{i}_0 + \sqrt{2}\mathbf{i}_1) \tag{3.8}$$

$$\mathbf{p}_{(3)} = э_2\mathbf{i}_2 + э_3\mathbf{i}_3, \quad \mathbf{p}_{(4)} = э_4\mathbf{i}_4 + э_5\mathbf{i}_5$$

The third and fourth components of expansion (3.7) when $n = 4$ and (3.8) are identical: $э_{(3)} \equiv \mathbf{p}_{(3)}$, $э_{(4)} \equiv \mathbf{p}_{(4)}$. The vectors $э_{(1)}$, $э_{(2)}$ and $\mathbf{p}_{(1)}$, $\mathbf{p}_{(2)}$ are the components of the expansion of the same strain vector in two orthogonal bases in the $\mathbf{i}_0$, $\mathbf{i}_1$ plane. These bases can be superposed by rotation through an angle $\zeta = \varphi + \operatorname{arctg}\frac{1}{\sqrt{2}}$. The linear relation

$$\boldsymbol{э}_{(1)} = \mathbf{p}_{(1)}\cos\zeta + \mathbf{p}_{(2)}\sin\zeta, \quad \boldsymbol{э}_{(2)} = -\mathbf{p}_{(1)}\sin\zeta + \mathbf{p}_{(2)}\cos\zeta$$

holds between the components $\ni_{(1)}$, $\ni_{(2)}$ and $\mathbf{p}_{(1)}$, $\mathbf{p}_{(2)}$ of expansions (3.7) when $n=4$ and (3.8). Unlike (3.7), expansion (3.8), which was introduced by Pobedrya, is solely determined by the type of symmetry of the material and is independent of the elastic constants of this material. However, the components of the spectral expansion (3.7) when $n=4$ appear in the simplest form of writing Hooke's law (3.6).

4. Extension of Il'yushin's particular postulate to anisotropic materials

According to Il'yushin's definition, a deformation trajectory with the corresponding stress vector constructed at each point in the trajectory constitutes an image of the process. It follows from expression (3.6) that, in the case of linearly elastic materials, the image of the process remains unchanged under orthogonal transformations (rotations and reflections) of the deformation trajectory in each characteristic subspace which is not one-dimensional. In particular, in the case of an isotropic material, this condition is satisfied in the five-dimensional deviator subspace. In the case of a cubic system, the condition for the isotropy of the images is satisfied in the plane of the vectors $\boldsymbol{\omega}_2$, $\boldsymbol{\omega}_3$ and in the three-dimensional subspace with the basis $\boldsymbol{\omega}_4$, $\boldsymbol{\omega}_5$, $\boldsymbol{\omega}_6$, in the case of a hexagonal (trigonal) material in the planes $\boldsymbol{\omega}_3$, $\boldsymbol{\omega}_4(\boldsymbol{\omega}_3, \boldsymbol{\omega}_5)$ and $\boldsymbol{\omega}_5$, $\boldsymbol{\omega}_6(\boldsymbol{\omega}_4, \boldsymbol{\omega}_6)$ and, in the case of a tetragonal material, in the single plane $\boldsymbol{\omega}_5$, $\boldsymbol{\omega}_6$.

The requirement, formulated by Il'yushin, that the images of processes are preserved was extended to non-linear constitutive relations for initially isotropic materials subject to small strains and was referred to by him as a particular postulate of isotropy.[1] According to this postulate, the images of processes located in the five-dimensional deviator subspace are not changed under orthogonal transformations of the deformation (loading) process. Numerous experiments have shown that the particular postulate is satisfied to a fairly high degree of accuracy in the case of small deformations. This is connected with the fact that, by virtue of Hooke's law, the particular postulate is always asymptotically satisfied in the initial (elastic) stage of the process.

An extension of the particular postulate to non-linear anisotropic materials can be formulated in the following way: *the images of processes located in the characteristic subspaces of the initial elasticity tensor corresponding to multiple eigenvalues are invariant under a group of characteristic orthogonal transformations*.

It follows from this generalization that the strain process in each non-unidimensional characteristic subspace is solely determined by the internal trajectory for the image of the deformation process in this subspace and is independent of the orientation of this trajectory with respect to the basis vectors of this subspace. If it is assumed that the deformation processes in the different characteristic subspaces take place independently of one another, then the representation of the relation between the strain and stress vectors in a characteristic subspace in the form

$$\boldsymbol{\sigma}_{(\alpha)} = \sum_{i=1}^{m} A^{i}_{(\alpha)} \mathbf{r}^{(\alpha)}_{i} \tag{4.1}$$

will be a corollary of the assertion which has been formulated. In relations (4.1), $A^i_{(\alpha)}[s_{(\alpha)}(\tau), \chi^1_{(\alpha)}(\tau), \ldots, \chi^{m-1}_{(\alpha)}(\tau), T(\tau)]^{\tau}_{\tau=t_0} = t$ is the functional of the deformation process $\ni_{(\alpha)}(t)$ and the functional of the law for the change in the temperature $T(t)$, $\mathbf{r}^{(\alpha)}_i$ is the basis of a characteristic subspace of dimension m, which is covariant with respect to characteristic orthogonal transformations, $s_{(\alpha)}$ is the length of an arc and $\chi_{(\alpha)}$ are the curvatures of the characteristic trajectory. In the case of smooth characteristic trajectories, it is convenient to take the Frenet basis, associated with the trajectory of the process $\ni_{(\alpha)}(t)$ as the basis $\mathbf{r}^{(\alpha)}_i$.

Relations (4.1) considerable constrain the reaction of anisotropic materials to a given deformation (loading) process which enables one to construct programs for the experimental definition of the relations determining the properties of a material within the framework of the limiting form of the particular postulate.

Starting out from representation (4.1), it is necessary for the specific definition of the properties of an initially anisotropic material to determine the orientation of the characteristic vectors, the characteristic subspaces and their dimensions on the basis of its elastic properties. By executing the deformation (loading) trajectories in each of the characteristic subspaces, it is possible to determine the form of the functionals $A^i_{(\alpha)}$.

As an example, consider a program for the specific definition of the properties of transversally-isotropic (hexagonal) materials in the case of simple isothermal deformation when all of the characteristic trajectories are radial. In this case, we have $\mathbf{r}^{(\alpha)}_i = \ni_{(\alpha)}/\ni_{(\alpha)}$ and the functionals $A^i_{(\alpha)}$ depend solely on the length of the characteristic deformation trajectory

$э_{(\alpha)}$. Relation (4.1) takes the form

$$\boldsymbol{\sigma}_{(\alpha)} = A_{(\alpha)}(\partial_{(\alpha)})\mathbf{э}_{(\alpha)}/\partial_{(\alpha)} \tag{4.2}$$

If the material is transversely-isotropic, that is, it belongs to the hexagonal system, relations (4.2) for each of the characteristic subspaces decompose into four independent relations containing four different material functions $A_{(1)(э_{(1)})}$, $A_{(2)(э_{(2)})}$, $A_{(3)(э_{(3)})}$, $A_{(4)(э_{(4)})}$. The first of these is determined from an experiment on triaxial deformation. If the axis of anisotropy coincides with the vector $\mathbf{e}_3$, and the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ lie in the plane of isotropy, then, in the experiment being considered, it is necessary to execute the process with a strain tensor $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_1(\mathbf{e}_1\mathbf{e}_1 + \mathbf{e}_2\mathbf{e}_2) + \boldsymbol{\varepsilon}_2\mathbf{e}_3\mathbf{e}_3$ with $\varepsilon_2/\varepsilon_1 = \sqrt{2}\,\mathrm{tg}\zeta$, where the parameter ζ is the angle through which a rotation superposes bases (3.7) when $n=4$ and (3.8). The second function is determined from an analogous experiment in which $\varepsilon_2/\varepsilon_1 = -\sqrt{2}\,\mathrm{ctg}\zeta$. The function $A_{(3)}(э_{(3)})$ is determined either from a biaxial extension - compression experiment in the plane of isotropy of the material with a strain tensor $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{e}_1\mathbf{e}_1 - \mathbf{e}_2\mathbf{e}_2)$ or from a shear experiment in this plane with a strain tensor $\boldsymbol{\varepsilon} = \boldsymbol{\gamma}(\mathbf{e}_1\mathbf{e}_2 + \mathbf{e}_2\mathbf{e}_1)$ or from a combination of these processes, subject to the condition that $\boldsymbol{\varepsilon}/\boldsymbol{\gamma} = \mathrm{const}$. The function $A_{(4)}(э_{(4)})$ can be determined from a shear experiment in a plane containing the axis of anisotropy with a strain tensor $\boldsymbol{\varepsilon} = \boldsymbol{\gamma}_1(\mathbf{e}_1\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_1) + \boldsymbol{\gamma}_2(\mathbf{e}_2\mathbf{e}_3 + \mathbf{e}_3\mathbf{e}_2)$ subject to the condition that $\gamma_2/\gamma_1 = \mathrm{const}$.

The principal difference between relations (4.1) and the version of the non-linear relations for anisotropic materials in Ref. 10 is the fact that, in writing down (4.1), it was assumed that the deformation processes in the different characteristic subspaces are independent; these relations describe the reaction of materials belonging to a narrower class than the general version of the relations proposed earlier in Ref. 10.

The use of the limiting form of the particular postulate enables us on the one hand to obtain the simplest versions of the constitutive relations and, on the other hand, to reveal the limits of their reliability. In the case of isotropic materials, the independence of the images of distortion and volume change processes is a consequence of the limiting form of the particular postulate. If, however, during a loading process, which is executed in the deviator space (in the case of simple shear, for example), the strain vector emerges from the deviator plane starting from a certain instant of time, it is necessary to complicate the limiting form of the postulate in order to take account of dilatational effects. A similar situation also arises in the case of anisotropic materials. If a component of the strain vector, which is orthogonal to a given subspace, appears during a loading process that is executed in a characteristic subspace, then it is necessary to extend the limiting form of the particular postulate which leads to the construction of more general versions of the constitutive relations.

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